Lecture 19

Ex: Show that the ellipsoid $3x^2+2y^2+z^2=9$ is tangent to the sphere $x^2+y^2+z^2-8x-6y-8z+24=0$ at $(1,1,2)$.

Sol: Let $F(x,y,z)=3x^2+2y^2+z^2$ and $G(x,y,z)=x^2+y^2+z^2-8x-6y-8z+24$. Then the ellipsoid is the level surface $F(x,y,z)=9$ and the sphere is the level surface $G(x,y,z)=0$. The level surface $F(x,y,z)=9$ intersects the level surface $G(x,y,z)=0$ tangentially at $(1,1,2)$ if they have the same tangent plane at that point, i.e., if $\nabla F(1,1,2) \,//\, \nabla G(1,1,2)$.

$\nabla F = \langle 6x, 4y, 2z \rangle$, $\nabla F(1,1,2) = \langle 6,4,4 \rangle$.

$\nabla G = \langle 2x-8, 2y-6, 2z-8 \rangle$, $\nabla G(1,1,2) = \langle -6,-4,-4 \rangle$.

So, since $\nabla F(1,1,2) = -\nabla G(1,1,2)$, the two level surfaces are tangent at $(1,1,2)$.

$\diamond$

Ex: Find an equation for the tangent line to the intersection of the hyperboloid $x^2-y^2+z^2=6$ and the sphere $x^2+y^2+z^2=14$ at the point $(1,2,3)$.

Sol: We can write the hyperboloid as a level surface $F(x,y,z)=6$ of $F(x,y,z)=x^2-y^2+z^2$ and the sphere as a level surface $G(x,y,z)=14$ of $G(x,y,z)=x^2+y^2+z^2$.

A normal vector to the hyperboloid at $(1,2,3)$ is $\nabla F(1,2,3)=\langle 2,-4,6\rangle$ $(\nabla F=\langle 2x,-2y,2z\rangle)$ and a normal vector to the sphere at $(1,2,3)$ is $\nabla G(1,2,3)=\langle 2,4,6\rangle$ $(\nabla G=\langle 2x,2y,2z\rangle)$. Since $\nabla F(1,2,3)$ and $\nabla G(1,2,3)$ are both perpendicular to the curve of intersection (the curve lies in both surfaces), a vector tangent to the intersection at $(1,2,3)$ is

$$\vec{v}=\nabla F(1,2,3)\times\nabla G(1,2,3)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -4 & 6\\ 2 & 4 & 6\end{vmatrix}=\langle -48,0,16\rangle$$

So, the tangent line to the intersection at $(1,2,3)=P$ is:

$$\vec{l}(t)=\vec{v}t+\vec{OP}=\langle -48t,0,16t\rangle+\langle 1,2,3\rangle.$$

14.7 - Maxima and Minima

<u>Def</u>: Let $f = f(x,y)$. A point (a,b) is called a

- <u>local minimum</u> if $f(a,b) \leq f(x,y)$ for all (x,y) near (a,b). $f(a,b)$ is a <u>local minimum value</u> of f
- <u>local maximum</u> if $f(a,b) \geq f(x,y)$ for all (x,y) near (a,b). $f(a,b)$ is a <u>local maximum value</u> of f

A local minimum/maximum is called an <u>absolute minimum/maximum</u> if the respective inequalities hold for all (x,y) in the domain of f.

How do we go about finding these points? In calc I, we looked for critical points of $f(x)$, i.e., x-values, a, such that $f'(a) = 0$. For us, a <u>critical point</u> is a point (a,b) such that $\nabla f(a,b) = \vec{0}$.

To classify the point as a min/max in calc I, we used the second derivative. Here we also have a second derivative test. First, the <u>Hessian</u> of f is

$$Hf(x,y) = \begin{pmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{pmatrix}.$$

Second Derivatives Test:

Suppose that f has continuous second partials near a point (a,b) such that $\nabla f(a,b)=\vec{0}$, i.e., (a,b) is a critical point of f. Let $D(a,b)=\det(Hf(a,b))$, then

- if $D(a,b)>0$ and $f_{xx}(a,b)>0$, then $f(a,b)$ is a local minimum value of f.
- if $D(a,b)>0$ and $f_{xx}(a,b)<0$, then $f(a,b)$ is a local maximum value of f.
- if $D(a,b)<0$ then (a,b) is neither a local minimum or local maximum. (a,b) is called a saddle point.
- if $D(a,b)=0$, the test is inconclusive.

Ex: Find and classify the critical points of

$$f(x,y)=y^3+3x^2y-6x^2-6y^2+2.$$

Sol: First $\nabla f=\langle 6xy-12x,\ 3y^2+3x^2-12y\rangle$

If $\nabla f=\vec{0} \Rightarrow \begin{cases} 6xy-12x=0 & ① \\ 3y^2+3x^2-12y=0 & ② \end{cases}$

$① \Rightarrow 6x(y-2)=0 \Rightarrow x=0 \text{ or } y=2$

If $x=0$, then ② reads:

$$0 = 3y^2 - 12y = 3y(y-4) \Rightarrow y = 0, 4$$

So, two critical points are $(0,0)$ and $(0,4)$.

If $y=2$, then ② reads:

$$0 = 12 + 3x^2 - 24 = 3x^2 - 12 \Rightarrow x = \pm 2$$

So, two more critical points are $(2,2)$ and $(-2,2)$.

Now $Hf = \begin{pmatrix} 6y-12 & 6x \\ 6x & 6y-12 \end{pmatrix}$, so $D = \begin{vmatrix} 6y-12 & 6x \\ 6x & 6y-12 \end{vmatrix}$

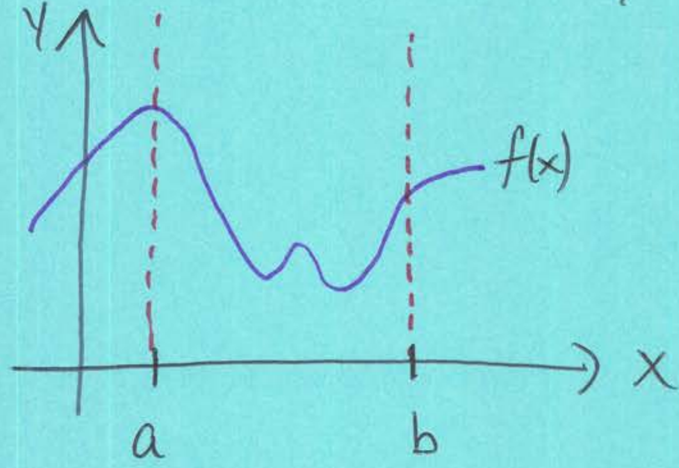
$$= (6y-12)^2 - 36x^2$$

Now we classify:

Crit. Pt.	f_{xx}	D	type
$(0,0)$	-12	144	max (local)
$(0,4)$	12	144	min (local)
$(2,2)$	0	-144	saddle
$(-2,2)$	-24	-144	saddle

$\diamond$

Back to calc I again: we could find the <u>absolute</u> max and min of a function $f:[a,b]\to\mathbb{R}$ by checking for critical values in (a,b) and also checking the boundary points. An illustrative example for why we check the boundary is:

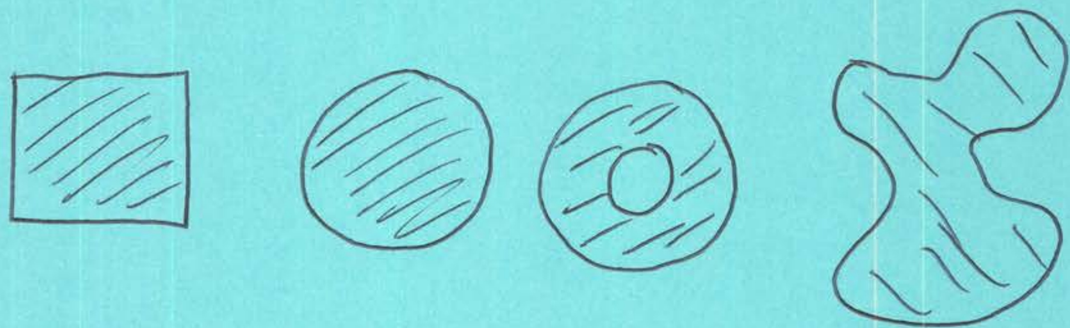


the absolute max occurs at $x=a$.

The main feature of this is that $[a,b]$ is a closed and bounded set.

To move this to our problem, we are asked to find absolute minimum and maximum values of a function defined on a closed and bounded set. So, the question is, "what does it mean for a set to be closed and bounded?"

<u>Closed sets</u> are sets which contain all of their boundary points. Intuitively, their boundary has no holes.

<u>Ex</u>: 

are closed.

<u>Non-ex</u>: 

are not closed.

A set is <u>bounded</u> in $\mathbb{R}^2$ if it can fit in a disk. The above examples are all bounded. An example of an unbounded set is:

$\{(x,y) \mid x \geq c,\ a \leq y \leq b\}$